CORRIGENDUM TO: SPIN h AND FURTHER GENERALISATIONS OF SPIN

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ABSTRACT. We qualify a claim made in [AM21], regarding the dimensions in which all orientable manifolds admit spin^h structures, with a compactness assumption, and comment on when this assumption can be removed.

In [AM21], the present authors made the following statement ([AM21, Theorem 1.3], [AM21, Corollary 3.10]): Every orientable manifold of dimension ≤ 7 is spin^h.

Here and throughout, we take all manifolds to be smooth. The argument for manifolds of dimension 6 and 7 invoked Cohen's immersion theorem [Coh85] in order to obtain a codimension 4 immersion in Euclidean space, followed by an application of [AM21, Proposition 3.9] that such an immersion guarantees the existence of a spin^h structure. However, in order to apply [Coh85], one should qualify the statement by assuming the manifolds in question are compact.

In the present corrigendum, we note that the following holds, using only (other) results of [AM21] and results preceding [Coh85]:

Theorem. *The following hold:*

- (1) Every (not necessarily compact) orientable manifold of dimension ≤ 5 is spin^h.
- (2) Compact orientable manifolds of dimension 6 and 7 are $spin^h$.
- (3) A non-compact orientable manifold M of dimension 6 or 7 is $spin^h$ if and only if $W_5(TM) = 0$.
- (4) A non-compact orientable manifold M of dimension 6 or 7 with no elements of order exactly four in H⁵(M;ℤ) is spin^h.

By "no elements of order exactly four" we mean that any $x \in H^5(M; \mathbb{Z})$ satisfying 4x = 0 also satisfies 2x = 0.

Proof. Part (1) is already proved in [AM21, p.5] without appealing to [Coh85] as a corollary of [AM21, Corollary 2.6], which states that the primary obstruction to the existence of a spin^h structure on an oriented manifold is the fifth integral Stiefel–Whitney class W_5 . This is an integral class of order two and hence vanishes on all orientable manifolds of dimension ≤ 5 . Here and throughout, we use the fact that the top cohomology (with any coefficients) of a non-compact manifold vanishes, e.g. see [Wh61, Lemma 2.1]; see also [Br62, Theorem 2] that every (not necessarily compact) manifold with boundary admits a collar neighborhood of its boundary, and hence it deformation retracts onto its interior.

That compact orientable six-manifolds immerse in \mathbb{R}^{10} is proved in [Hir61, Corollary 9], and hence they admit spin^h structures by [AM21, Proposition 3.9]. Note that the statement of [Hir61, Corollary 9] does not include compactness, though it is clear from the proof that the manifold is assumed to be closed; the general compact case then follows by taking the double if the boundary is non-empty. For compact orientable seven-manifolds M, we use the result listed in the second table of [AtDu72, p.25] (see also the footnote (1) in loc. cit.), that the only obstruction to a compact seven-manifold admitting three linearly independent vector fields is the integral Bockstein of w_4 , i.e. W_5 . Again, the result is stated for closed manifolds, and the compact-with-boundary case follows by considering the double. (Note, for orientable seven-manifolds, W_5 is a priori the single obstruction to finding three linearly independent sections over the five-skeleton.) This class vanishes by [Mas62, Theorem 3]. Hence the tangent bundle of M splits off a trivial rank three bundle, giving an orientable rank four bundle with the same w_2 as TM, and we again apply [AM21, Proposition 3.9]. This proves part (2).

Now let M be a non-compact orientable six-manifold. Since $H^6(M; \mathbb{Z}) = 0$, there are no secondary or higher obstructions to admitting a spin^h structure beyond W_5 ; this establishes part (3) for sixmanifolds. Choose an increasing exhaustion $\{M_i\}$ by compact manifolds with boundary. For an abelian group A and integer k > 1, we have the short exact Milnor sequence [Sw17, Proposition 7.66]

$$0 \to \varprojlim^{1} H^{k-1}(M_i; A) \to H^k(M; A) \to \varprojlim^{1} H^k(M_i; A) \to 0.$$

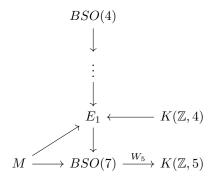
From the long exact sequence in cohomology associated to the short exact coefficient sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2 \to 0$$

we have the following commutative diagram:

For each M_i , we have $W_5(M_i) = 0$ by taking the double and applying [Mas62, Theorem 2] (or crossing the double with a circle and applying [Mas62, Theorem 3] again). Generally for an orientable manifold, the mod 2 reduction of W_5 is w_5 . From here and by naturality, $w_5(M) \in H^5(M; \mathbb{Z}/2)$ maps to the zero element in $\lim_{\to} H^5(M_i; \mathbb{Z}/2)$. By [MiSt74, Lemma 10.3], the term $\lim_{\to} H^4(M; \mathbb{Z}/2)$ vanishes. Therefore $w_5(M)$ must be zero as well. Now, $W_5(M) \in H^5(M; \mathbb{Z})$ is an element of order two which maps to $w_5(M) = 0$ by mod 2 reduction. Therefore it is in the image of the map $H^5(M; \mathbb{Z}) \xrightarrow{.2} H^5(M; \mathbb{Z})$. Since by assumption there are no elements of order exactly four in $H^5(M; \mathbb{Z})$, it follows that $W_5(M)$ must be the zero class. This establishes part (4) for six-manifolds.

Now let M be an orientable non-compact seven-manifold. We will show that the secondary obstruction to the existence of a spin^h structure vanishes, establishing parts (3) and (4). Take an exhaustion $\{M_i\}$ by compact seven-manifolds with boundary. If $W_5(M) = 0$, which will for instance be true given the torsion condition on $H^5(M;\mathbb{Z})$ by the argument above, we can choose a lift of the classifying map of the tangent bundle $M \to BSO(7)$ to E_1 , the second stage of the relative Postnikov tower of $BSO(4) \to BSO(7)$,

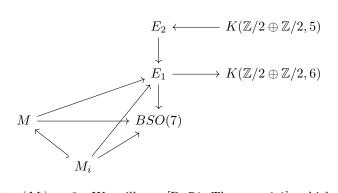


Restricting to the M_i gives a compatible system of lifts to E_1 . We consider now the secondary obstruction $\mathfrak{o}(M)$ to admitting three linearly independent vector fields. This is a class in $H^6(M; \pi_5(V(3,7)))$, where V(3,7) is the Stiefel manifold of 3-frames in \mathbb{R}^7 . We have the following exact sequence of homotopy groups:

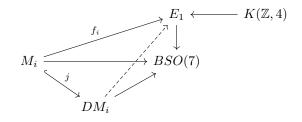
$$\pi_6(BSO(7)) \to \pi_5(V(3,7)) \to \pi_5(BSO(4)) \to \pi_5(BSO(7)).$$

The natural map $BSO(7) \to BSO$ is an isomorphism on $\pi_{\leq 6}$, and hence we have $\pi_5(BSO(7)) = \pi_6(BSO(7)) = 0$. Furthermore, $\pi_5(BSO(4)) \cong \pi_4(SO(4)) \cong \pi_4(S^3 \times S^3) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Since we are fixing the lifts $M \to E_1$ and $M_i \to E_1$, and they are compatible, the secondary obstruction to lifting further to E_2 is natural, i.e. $\mathfrak{o}(M_i)$ is the restriction of $\mathfrak{o}(M)$.



Let us now argue that $\mathfrak{o}(M_i) = 0$. We will use [Du74, Theorem 1.1], which gives us that for any choice of lift to E_1 on a *closed* orientable seven-manifold, the secondary obstruction vanishes. In order to apply this to M_i , we consider the double DM_i . We will argue that the lift $M_i \xrightarrow{f_i} E_1$ (obtained by restricting $M \xrightarrow{f} E_1$) extends to a lift $DM_i \to E_1$. Then, by applying loc. cit., we will have $\mathfrak{o}(DM_i) = 0$ and hence $\mathfrak{o}(M_i) = 0$.



First, choose any lift $DM_i \xrightarrow{G} E_1$ of $DM_i \rightarrow BSO(7)$; this exists since W_5 vanishes on any closed orientable seven-manifold. Now, f_i and the restriction of G to M_i differ by the action of an element

x in $[M_i, K(\mathbb{Z}, 4)] = H^4(M_i; \mathbb{Z})$ (this group acts simply transitively on the homotopy classes of lifts to E_1). Let us denote this by $[f_i] = x \cdot [G|_{M_i}]$.

Now observe that x is the restriction of a class $X \in H^4(DM_i; \mathbb{Z})$. Namely, consider the Mayer–Vietoris sequence for the double:

$$\cdots \to H^4(DM_i;\mathbb{Z}) \to H^4(M_i;\mathbb{Z}) \oplus H^4(M_i;\mathbb{Z}) \to H^4(\partial M_i;\mathbb{Z}) \to \cdots$$

The element (x, x) maps to zero, and hence $x = j^* X$ for some $X \in H^4(DM_i; \mathbb{Z})$.

Therefore, if we consider the (class of the) lift $X \cdot [G]$ on DM_i instead of [G], by naturality we have that its restriction to M_i is $x \cdot [G|_{M_i}]$, i.e. $[f_i]$.

Now we have that $\mathfrak{o}(M_i) = 0$ for all *i*. Consider the short exact sequence

 $0 \to \underline{\lim}^{1} H^{3}(M_{i}; \mathbb{Z}/2 \oplus \mathbb{Z}/2) \to H^{4}(M; \mathbb{Z}/2 \oplus \mathbb{Z}/2) \to \underline{\lim} H^{4}(M; \mathbb{Z}/2 \oplus \mathbb{Z}/2) \to 0.$

Since $H^*(-; \mathbb{Z}/2 \oplus \mathbb{Z}/2)$ is naturally isomorphic to $H^*(-; \mathbb{Z}/2) \oplus H^*(-; \mathbb{Z}/2)$, the $\lim_{i \to 1}^{1}$ term vanishes. Further, since $\mathfrak{o}(M)$ maps to $(\mathfrak{o}(M_i))_i$, which is the zero element, by injectivity we have that $\mathfrak{o}(M) = 0$. Since M has the homotopy type of a six-complex, the secondary obstruction is also the final obstruction to admitting three linearly independent vector fields, and we conclude that M admits a spin^h structure.

Likewise, the statement in the paragraph preceding [AM21, Remark 3.5], that every orientable n-manifold is spin^{$n-\alpha(n)$} (where $\alpha(n)$ is the number of one's in the binary expansion of n) should be qualified with a compactness assumption. Removing the compactness assumption, we can of course appeal to Whitney's immersion theorem to conclude that every orientable n-manifold is spinⁿ⁻¹. Similarly, compactness should be assumed where appropriate in [AM21, Section 5].

- **Remark.** (1) We record also that there are some inaccuracies in the table of homotopy groups of Stiefel manifolds in [EDM93, p.1747]. Namely, taking n = 4 and m = 3, 8s 1, 8s + 3, one sees from the long exact sequence in homotopy groups, as above, that $\pi_5(V(m, m + 4))$ in the notation of loc. cit. this is $\pi_5(V(m + 4, m))$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, contrary to what is listed therein.
 - (2) The primary obstruction to a spin^c structure on an orientable manifold is W_3 , and compact orientable four-manifolds are spin^c. An analogous argument to the above then shows that non-compact orientable four-manifolds with no elements of order exactly four in $H^3(-;\mathbb{Z})$ are spin^c. The four-torsion assumption can be removed in this case [TV], and it is not clear whether one should expect this in the theorem above.

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